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LETTER TO THE EDITOR

On the complete solution of the Hirota–Satsuma system through the 'dressing' operator technique

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Abstract. We have applied the 'dressing' operator method of Zakharov *et al* for obtaining the complete solution of the Hirota–Satsuma coupled system. This method helps us essentially to solve the inverse problem associated with a fourth-order differential operator. Also the method of the 'dressing' operator has the distinct advantage of generating solutions other than soliton by varying the boundary condition in an appropriate fashion.

In recent times there have been several attempts to perform inverse spectral analysis of the higher order differential operator, for solving a wider class of nonlinear equation. Among the successful attempts we can mention those of Caudrey (1982), Kaup (1980) and Dieft *et al* (1982). However, each of these papers deals with a third-order eigenvalue problem. We here proceed for the exact solution of the coupled system of Hirota and Satsuma (1981), which is shown to be completely integrable with the help of a fourth-order eigenvalue problem.

The associated time dependence is governed by a third-order operator. Since the formulation of the inverse problem for the fourth-order operator is quite complicated we have followed a different route by applying the dressing operator technique of Zakharov *et al* to the bare operators. The methodology yields solutions more general than the inverse scattering technique according to the nature of the imposed boundary condition.

The equations under consideration read:

$$U_t - a(U_{3x} + 6UU_x) = 2b\phi\phi_x, \qquad \phi_t + \phi_{3x} - 3U\phi_x = 0. \tag{1}$$

It has been shown recently that this system admits a Lax representation involving a fourth-order eigenvalue operator L and a third-order time evolution operator M, written as:

$$\begin{aligned} L &= -\partial_x^4 + 2U\partial_x^2 + 2(U_x - \phi_x)\partial_x + U_{2x} - \phi_{2x} + U^2 - \phi^2 \\ M &= -2(\partial_x^3 + \tfrac{3}{2}U\partial_x + \tfrac{4}{3}U_x - \tfrac{3}{2}\phi_x). \end{aligned} \tag{2}$$

So an effective construction of solutions belonging to the soliton, breather or any other class can be possible only with the help of an IST applied to $L\psi = \lambda\psi$ and adjoining the time evolution $\psi_t = M\psi$. But as we noted earlier the formulation of IST even for a third-order operator is not straightforward. So we started from the bare operators $L^0 = -\partial_x^4$ and $M^0 = -2\partial_x^3$ and followed the procedure of dressing up the bare operators to construct those mentioned in equation (2) along with the explicit solutions of ϕ

and U in terms of kernels obeying linear differential equations. In practice we usually deal with the Volterra kernels K_+ and K_-.

To proceed with the formulation we first set up some notations. The bare operators are denoted by 'zero' subscripts. In our particular case we denote:

$$\hat{L}_0^l = l_1\partial_x^4, \quad l_1 = 1 \qquad \hat{L}_0^2 = l_2\partial_x^3, \quad l_2 = -2$$

and define:

$$\begin{aligned}
&\hat{\hat{M}}_1 = \alpha\partial_t + \hat{L}_1\\
&\hat{L}_1 = \partial_x^4 + 2U\partial_x^2 + 2(U_x - \phi_x)\partial_x + U_{2x} - \phi_{2x} + U^2 - \phi^2\\
&\hat{M}_1 = \alpha\partial_t + \hat{L}_0^1
\end{aligned} \tag{3}$$

where we have also set $\hat{M}_2 = \hat{L}_0^2$, $\hat{\hat{M}}_2 = \hat{L}^2$. Now the basic technique in the construction of the dressed up operator from the bare one is to use the Volterra triangular operator K_+ and K_- which are taken to be factors of F:

$$1 + F = (1 + K_+)^{-1}(1 + K_-). \tag{4}$$

In this section and the following we have exclusively used the notation of the excellent article by Manin (1978). Operations of $\hat{F}$ and $\hat{K}$ are defined through:

$$\begin{aligned}
&\hat{F}\psi = \int_{-\infty}^{\infty} F(xz)\psi(z)\,\mathrm{d}z\\
&\hat{K}\psi = \int_{x}^{\infty} K(xy)\psi(y)\,\mathrm{d}y.
\end{aligned} \tag{5}$$

It is easy to demonstrate that F and K satisfy the Gelfand–Levitan equation:

$$K(xy) + F(xy) + \int_x^{\infty} K(xs)F(sy)\,\mathrm{d}s = 0. \tag{6}$$

With the help of the basic theorems of the dressing operators we construct first the differential equations satisfied by F and K.

Let us start by considering ψ to be a simultaneous eigenfunction of

$$[\hat{M}_1, \hat{F}]\psi = 0 \qquad \text{and} \qquad [\hat{M}_2\hat{F}]\psi = 0. \tag{7}$$

Then using the form of $\hat{M}_1$ we get

$$\alpha\partial_t\hat{F}\psi + \hat{L}_0^1\hat{F}\psi - \hat{F}(\alpha\partial_t + \hat{L}_0^1)\psi = 0.$$

Using $\hat{L}_0^1 = \partial_x^4$ we have

$$\alpha\partial_t\int_{-\infty}^{\infty} F(xz)\psi(z)\,\mathrm{d}z + \hat{L}_0^{(1)}\int_{-\infty}^{\infty} F(xz)\psi(z)\,\mathrm{d}z$$
$$-\int_{-\infty}^{\infty} F(xz)[\alpha\partial\psi/\partial t + \hat{L}_0^{(1)}(z)\psi(z)]\,\mathrm{d}z = 0.$$

Integrating by parts we get:

$$\int_{-\infty}^{\infty}\left(\alpha\frac{\partial F}{\partial t} + \frac{\partial^4 F(xz)}{\partial x^4} - \frac{\partial^4 F(xz)}{\partial z^4}\right)\psi(z)\,\mathrm{d}z = 0$$

which implies:

$$\alpha \frac{\partial F}{\partial t} + \frac{\partial^4 F(xz)}{\partial x^4} - \frac{\partial^4 F(xz)}{\partial z^4} = 0. \tag{8}$$

Similarly from the second equation of (7) we get:

$$-2\partial_x^3 \int_{-\infty}^{\infty} F(xz)\psi(z)\,\mathrm{d}z + 2\int_{-\infty}^{\infty} F(xz)\frac{\partial^3\psi(z)}{\partial z^3}\,\mathrm{d}z = 0$$

or

$$\int_{-\infty}^{\infty} \left(\frac{\partial^3 F(xz)}{\partial x^3} + \frac{\partial^3 F(xz)}{\partial z^3}\right)\psi(z)\,\mathrm{d}z = 0$$

which implies

$$\partial^3 F(xz)/\partial x^3 + \partial^3 F(xz)/\partial z^3 = 0. \tag{9}$$

(8) and (9) are the basic equations satisfied by the operator. These solutions used in the Gelfand–Levitan equation (6) yield $K(xy)$ which in turn is to be connected to the nonlinear fields.

The basic equations of operator dressing are:

$$\begin{aligned}(1+\hat{\hat{M}}_1)(1+\hat{K}_+)\psi &= (1+\hat{K}_+)(1+\hat{M}_1)\psi \\ (1+\hat{\hat{M}}_2)(1+\hat{K}_+)\psi &= (1+\hat{K}_+)(1+\hat{M}_2)\psi.\end{aligned} \tag{10}$$

Writing out the first equation of (10) we get:

$$\begin{aligned}\alpha \int_x^{\infty} \frac{\partial K}{\partial t}(xz)\psi(z)\,\mathrm{d}z &+ \partial_x^4 \int_x^{\infty} K(xz)\psi(z)\,\mathrm{d}z + 2U\psi_2 \\ &+ 2U\partial_x^2 \int_x^{\infty} K(xz)\psi(z)\,\mathrm{d}z + 2(U_x - \phi_x)\psi_1 \\ &+ 2(U_x - \phi_x)\partial x \int_x^{\infty} K(xz)\psi(z)\,\mathrm{d}z \\ &+ (U_{2x} - \phi_{2x})\psi + (U^2 - \phi^2)\psi + (U_{2x} - \phi_{2x} + U^2 - \phi^2) \\ &\times \int_x^{\infty} K(xz)\psi(z)\,\mathrm{d}z = \int_x^{\infty} K(xz)\partial_z^4\psi(z)\,\mathrm{d}z.\end{aligned} \tag{11}$$

Now making extensive use of the following formulae:

$$\begin{aligned}\partial_x^n \int_x^{\infty} K(xz)\psi(z)\,\mathrm{d}z &= \int_x^{\infty} \partial_x^n K(xz)\psi(z)\,\mathrm{d}z \\ &- \sum_{i=0}^{n-1} \partial_x^i\{\partial_x^{n-1-i} K(xz)|_{z=x}\,\psi(x)\}\end{aligned}$$

and:

$$\begin{aligned}\int_x^{\infty} K(xz)\partial_z^n\psi(z)\,\mathrm{d}z &= (-1)^n \int_x^{\infty} \partial_z^n K(xz)\psi(z)\,\mathrm{d}z \\ &+ \sum_{i=0}^{n-1} (-1)^{i+1}[\partial_z^i K(xz)]\partial_x^{n-1-i}\psi(x),\end{aligned} \tag{12}$$

we get the following equations connecting the nonlinear fields U, ϕ to the kernel $K(xz)$ and also the equation satisfied by K.

$$U = 2\mathrm{d}(K(xx))/\mathrm{d}x \tag{13}$$

$$\begin{aligned}\phi_x = \tfrac{1}{2}[\partial_z^2 K(xz) - \partial_x^2 K(xz)]_{z=x} + \tfrac{1}{2}\,\mathrm{d}^2 K(xx)/\mathrm{d}x^2 \\ - \partial_x[(\partial_z K(xz))_{z=x}] - 2K(xx)\,\mathrm{d}K(xx)/\mathrm{d}x\end{aligned} \tag{14}$$

$$\begin{aligned}-[\partial_x^3 K(xz)]_{z=x} - 2U(\partial_x K(xz))_{z=x} + 2(\phi_x - U_x)K(xx) \\ + U_{2x} - \phi_{2x} + U^2 = \phi^2 = \partial_z^3 K(xz)|_{z=x}.\end{aligned} \tag{15}$$

An effective way of constructing an explicit solution is to solve equations (8) and (9) which after the change of variables $\omega = x + z$ and $\nu = x - z$, are transformed to the following form:

$$(\partial/\partial\omega)(\partial^2 F/\partial\omega^2 + 3\partial^2 F/\partial\nu^2) = 0 \tag{16}$$

$$\alpha\partial F/\partial t + 8(\partial^2/\partial\omega\partial\nu)(\partial^2 F/\partial\omega^2 + \partial^2 F/\partial\nu^2) = 0.$$

The first equation of (16) can be integrated in several ways by choosing different constants of integration.

We here choose the simplest one and consider only that class of solutions for which

$$\partial^2 F/\partial\omega^2 + 3\partial^2 F/\partial\nu^2 = 0.$$

A general solution of this equation can be written as:

$$F = \Phi(\nu + \mathrm{i}\omega\sqrt{3}) + \Psi(\nu - \mathrm{i}\omega\sqrt{3}) \tag{17}$$

where the dependence of the time variable is still to be fixed by the second equation of (16). Substituting this expression in the second equation (16) (here we have performed the calculation with the first term of (17) but a similar calculation can be done with the second term also), we obtain

$$\alpha\partial\Phi/\partial t - 16\mathrm{i}v_3\,\partial^4\Phi/\partial\eta^4 = 0 \tag{18}$$

where

$$\eta = \nu + \mathrm{i}\omega\sqrt{3}.$$

The most general solution of equation (18) is

$$\Phi(\eta, t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-(x^2+y^2)}\Lambda(\eta + 2\sqrt{2}y\sqrt{x}\lambda' t^{1/4})\,\mathrm{d}x\,\mathrm{d}y \tag{19}$$

where Λ is an arbitrary function.

Instead of discussing the most general situation we here consider a special case of (19) which yields:

$$\Phi = \cosh(\nu + \mathrm{i}\omega\sqrt{3})\exp(16\mathrm{i}\sqrt{3}t/\alpha). \tag{20}$$

Using (17) and (20) the Gelfand–Levitan equation can be recast into the form

$$\cosh(2\omega' z - 2\omega'^2 x)\exp(16\mathrm{i}\sqrt{3}t/\alpha) + K(xz)$$
$$+ \int_x^{\infty} K(xs)\cosh(2\omega' z - 2\omega_s'^2)\exp(16\mathrm{i}\sqrt{3}t/\alpha)\,\mathrm{d}s = 0.$$

We seek a solution of K in the form:

$$K(xz) = \exp(16\mathrm{i}\sqrt{3}t/\alpha)\sum(xzt)$$
$$= \exp(-16\mathrm{i}\sqrt{3}t/\alpha)[A(xt)\cosh(2\omega' z - 2\omega_x'^2)$$
$$+ B(xt)\sinh(2\omega' z - 2\omega_x'^2)] \tag{22}$$

which when used in (21) yields

$$A = (1/q)[1 + \exp(16\mathrm{i}\sqrt{3}t/\alpha)(\omega'^2/2\mathrm{i}\sqrt{3})\sinh(2\mathrm{i}\sqrt{3}x)]$$
$$B = -(1/q)(\omega'^2/2\mathrm{i}\sqrt{3})\exp(16\mathrm{i}\sqrt{3}t/\alpha)\cosh(2\mathrm{i}\sqrt{3}x)$$
$$\omega'^3 = 1$$
$$q = 1 - \frac{1}{(2\mathrm{i}\sqrt{3})^{-1}}\exp\left(\frac{16\mathrm{i}\sqrt{3}t}{\alpha}\right)\sinh(2\mathrm{i}\sqrt{3}x) - (1/12)\exp\left(\frac{32\mathrm{i}\sqrt{3}t}{\alpha}\right). \tag{23}$$

In our above expressions ω' is the cube root of unity. It is interesting to observe that if we search for separable solutions of equation (18) then the biharmonic equation that results for the spatial part possesses both hyperbolic and sinusoidal type solutions. In equations (19), (20) and (21) we have chosen only one type of solution. Plugging these solutions for $K(xz)$ in (13) and (14) we can now obtain the nonlinear fields $U(xt)$ and $\phi(xt)$. Also it is interesting to observe that equation (15) is identically satisfied by any particular set of solutions for U, ϕ and K. Though our displayed solution for A, B involves hyperbolic functions, since their arguments are imaginary, they are actually sinusoidal functions. These types of rational solutions for U and ϕ which involve these two types of functions are not usually obtainable through the usual procedure of inverse scattering transform. Furthermore it should be noted that by considering exponential solutions of $F(xy)$ we can generate soliton or multisoliton solutions. For example let us consider F in the form:

$$F(xz) = A\exp\left(-\frac{R}{\lambda - 1}(\lambda x - z) + \frac{4Rt}{\alpha(1-\lambda)^3}\right) \tag{24}$$

which yields:

$$K(xz) = -\left[A\exp\left(\frac{-R}{\lambda-1}(\lambda x - z)\right)\right]\Big/\left[\frac{A}{R}\mathrm{e}^{-Rx} + \exp\left(\frac{-4Rt}{\alpha(1-\lambda)^3}\right)\right] \tag{25}$$

implying the following structure for U and ϕ:

$$U = (R^2/2)\,\mathrm{sech}^2(\beta t + Rx - \log A/R)$$

which asymptotically goes to zero as $x \to \infty$. On the other hand the other nonlinear field ϕ is given as:

$$\phi = (5R^2/8)\,\mathrm{sech}^2\left(Rx + \beta t - \log\frac{A}{R}\right) - \frac{\lambda R^2}{(\lambda - 1)}$$
$$\times\left[\mathrm{sech}\left(Rx + \beta t - \log\frac{A}{R}\right) - \tanh\left(Rx + \beta t - \log\frac{A}{R}\right)\right].$$

So that as $x \to \infty$, $\phi \to -\lambda R^2/(\lambda - 1)$ (a constant) and these two asymptotic values of U and ϕ are consistent with the nonlinear equation (1). At this point we can mention that by adjusting the values of the constants R, β, A and λ it is possible to manufacture a solution which is simply proportional to tanh ξ, tending to a constant value for large values of x and t.

In our above calculation we have demonstrated that it is possible to obtain explicit soliton or rational solutions for the Hirota–Satsuma system via the dressing operator formalism. The procedure is actually an 'inversion' of the fourth-order differential operator without actually using the 'scattering data'.

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